Suggested solutions to the Contract Theory resit exam on February 17, 2011
VERSION: 11 March 2011

Question 1

There are two countries, A (which is rich) and B (which is poor). A monopoly firm can produce and sell its good — a particular computer program — in the two countries. The program can be offered in different qualities. The payoff of a representative consumer in country $i \in \{A, B\}$, if consuming the program at price p when the quality is q, is given by

$$U_i = \theta_i q - p,$$

where it is assumed that

$$\theta_A > \theta_B > 0.$$

The program can be offered at different prices and different qualities in the two countries. However, a consumer can, without incurring any transaction costs, choose to purchase the program in the country in which he is not living, if he prefers that price-quality combination. Therefore, if the firm offers the good in both countries, the following two incentive compatibility constraints must be satisfied:

$$\theta_A q_A - p_A \geq \theta_A q_B - p_B \tag{IC-A}$$

and

$$\theta_B q_B - p_B \geq \theta_B q_A - p_A. \tag{IC-B}$$

In addition, if the firm wants the consumers in both countries to actually purchase the program, their individual rationality constraints must be satisfied. Assuming that all consumers' outside option yields the payoff zero, these constraints can be written as

$$\theta_A q_A - p_A \geq 0 \tag{IR-A}$$

and

$$\theta_B q_B - p_B \geq 0. \tag{IR-B}$$

In addition, in country B there is, for any given quality of the program, a maximum price that the firm is allowed to charge. The level of the price cap is linear in the quality and it can be written as

$$p_B \leq d_B q_B, \tag{PC-B}$$

where

$$0 < d_B < \theta_B.$$

The monopoly firm has a quadratic cost function and wants to maximize its profits. These profits can be written as

$$V = \nu \left(p_A - \frac{c}{2} q_A^2 \right) + (1-\nu) \left(p_B - \frac{c}{2} q_B^2 \right)$$

where $c > 0$ is a parameter, $\nu \in (0,1)$ is the number of consumers in country A, and $(1-\nu)$ is the number of consumers in country B.

a) **Let the first-best levels of q_A and q_B be defined as the ones that maximize the total surplus,**

$$(1-\nu)\,\theta_B q_B + \nu \theta_A q_A - (1-\nu)\,\frac{c}{2} q_B^2 - \nu \frac{c}{2} q_A^2.$$

Calculate these first-best levels. Explain the economic intuition behind your result.

- According to the question, the first-best levels of q_A and q_B are defined as the ones that maximize the total surplus,

$$TS = (1-\nu)\,\theta_B q_B + \nu \theta_A q_A - (1-\nu)\,\frac{c}{2} q_B^2 - \nu \frac{c}{2} q_A^2.$$

Differentiating this expression with respect to q_A yields

$$\frac{\partial TS}{\partial q_A} = \nu \theta_A - \nu c q_A.$$

This expression is strictly positive evaluated at $q_A = 0$, so a corner solution is clearly not optimal. Instead the value of q_A that maximizes TS can be found by setting the derivative equal to zero (as TS is strictly concave in q_A):

$$\frac{\partial TS}{\partial q_A} = \nu \theta_A - \nu c q_A = 0 \Rightarrow q_A^{FB} = \frac{\theta_A}{c}.$$

Similarly, for q_B we have

$$\frac{\partial TS}{\partial q_B} = (1-\nu)\,\theta_B - (1-\nu)\,c q_B = 0 \Rightarrow q_B^{FB} = \frac{\theta_B}{c}.$$

- The intuition: The total surplus is defined as the total sum of net surplus that is generated by the production and consumption of the good. The first-best level of q_A, for example, maximizes that net surplus; therefore it trades off the benefits that the consumers in country A enjoy by consuming q_A against the firm's cost of producing that quantity. At the optimum the marginal benefit, $\nu\theta_A$, must equal the marginal production cost, $\nu c q_A$, which yields the first-best level q_A^{FB}.

b) **Return to the model with asymmetric information described above and solve for the optimal second-best prices and qualities. Assume that the parameters of the model are such that the monopoly firm optimally sells in both countries. Explain how the optimal second-best qualities differ from the optimal first-best qualities. Also explain the economic intuition behind any differences. Do the consumers of any one of the countries get any rents at the second-best optimum? If so, which country or which countries? Explain why.**

- There is a price cap in country B (the one that is poorer, or with a lower valuation).
- We are supposed to assume that the parameters of the model are such that the monopoly firm optimally sells in both countries. The arguments below therefore take as given that $q_A > 0$ and $q_B > 0$. In addition, it is taken as given that the monopoly firm does want to offer the same contract to the two consumer groups (bunching). In the course we showed that in very similar two-type models of adverse selection, bunching is not optimal.
- The firm's objective:

$$V(q_A, q_B, p_B, p_A) = \nu \left(p_A - \frac{c}{2}q_A^2\right) + (1-\nu)\left(p_B - \frac{c}{2}q_B^2\right).$$

- Constraints:
  - The B customers must prefer their bundle to no bundle at all:
$$\theta_B q_B - p_B \geq 0. \tag{IR-B}$$
  - The A customers must prefer their bundle to no bundle at all:
$$\theta_A q_A - p_A \geq 0. \tag{IR-A}$$
  - The B customers must prefer their bundle to the A customers' bundle:
$$\theta_B q_B - p_B \geq \theta_B q_A - p_A. \tag{IC-B}$$
  - The A customers must prefer their bundle to the B customers' bundle:
$$\theta_A q_A - p_A \geq \theta_A q_B - p_B. \tag{IC-A}$$
  - And the new price ceiling constraint:
$$p_B \leq d_B q_B. \tag{PC-B}$$

- Given the assumption that $0 < d_B < \theta_B$, PC-B implies IR-B. To see that, write

$$p_B \overbrace{\leq}^{\text{By PC-B}} d_B q_B \overbrace{<}^{\text{By } d_B<\theta_B \text{ and } q_B>0} \theta_B q_B.$$

This series of inequalities says that $p_B < \theta_B q_B$, which is indeed stronger than IR-B (implying that the B consumers will get a rent). We can thus ignore IR-B.

- Moreover, we have:

$$\theta_A q_A - p_A \overbrace{\geq}^{\text{By IC-A}} \theta_A q_B - p_B \overbrace{>}^{\text{By } \theta_A>\theta_B \text{ and } q_B>0} \theta_B q_B - p_B$$
$$\overbrace{>}^{\text{By } \theta_B>d_B \text{ and } q_B>0} d_B q_B - p_B \overbrace{\geq}^{\text{By PC-B}} 0.$$

This means that if PC-B and IC-A are satisfied, then so is IR-A (indeed, with a strict inequality, which means that also the A consumers will get a rent). We can thus ignore IR-A.

- We can also guess that IC-B is satisfied at the optimum (and check afterwards).

- The remaining constraints are now

$$p_B \leq d_B q_B. \tag{PC-B}$$

$$\theta_A q_A - p_A \geq \theta_A q_B - p_B. \tag{IC-A}$$

- Claim: Both constraints must bind at the optimum.

  - *Proof that PC-B binds (the argument for IC-A is similar):* Suppose PC-B did not bind at the optimum. Then p_B could be increased without violating PC-B or IC-A, which would increase the objective. This contradicts the assumption that we were at the optimum. Hence PC-B must bind at the optimum.

- Changing the inequality signs to equalities in the two constraints and then solving for p_A and p_B yields

$$p_B = d_B q_B, \tag{1}$$

$$p_A = \theta_A q_A - \theta_A q_B + p_B = \theta_A q_A - (\theta_A - d_B)\, q_B. \tag{2}$$

- Plugging p_A and p_B into the objective function we get

$$V(q_A, q_B) = \nu \left[\theta_A q_A - (\theta_A - d_B)\, q_B - \frac{c}{2} q_A^2\right] + (1-\nu)\left(d_B q_B - \frac{c}{2} q_B^2\right).$$

- FOC w.r.t. q_A:

$$\frac{\partial V(q_A, q_B)}{\partial q_A} = \nu [\theta_A - c q_A] = 0 \Rightarrow \boxed{q_A^{SB} = \tfrac{\theta_A}{c}.}$$

This means that the A consumers second best quantity is the same as under first best, $\boxed{q_A^{SB} = q_A^{FB}}$: there is "efficiency at the top".

- FOC w.r.t. q_B:

$$\frac{\partial V(q_A, q_B)}{\partial q_A} = -\nu (\theta_A - d_B) + (1-\nu)(d_B - c q_B) = 0 \Rightarrow \boxed{q_B^{SB} = \tfrac{d_B - \nu\theta_A}{(1-\nu)c}.}$$

  - Note that we have $\boxed{q_B^{SB} < q_B^{FB}}$:

$$\begin{aligned} q_B^{SB} &< q_B^{FB} \Leftrightarrow \frac{d_B - \nu\theta_A}{(1-\nu)c} < \frac{\theta_B}{c} \Leftrightarrow d_B - \nu\theta_A < (1-\nu)\theta_B \\ &\Leftrightarrow (\theta_B - d_B) + \nu(\theta_A - \theta_B) > 0, \end{aligned}$$

  which is always satisfied given the assumptions that $0 < d_B < \theta_B < \theta_A$ and $\nu \in (0,1)$. That is, the B consumers' quantity is distorted downwards relative to the first best.
  - Also note that since we assumed that the parameters of the model are such that $q_B^{SB} > 0$ (see the discussion above), we then implicitly assumed that $\boxed{d_B > \nu\theta_A}$.
  - Finally note that we have $\boxed{q_B^{SB} < q_A^{SB}}$ (which we need in a proof below):

$$q_B^{SB} < q_A^{SB} \Leftrightarrow \frac{d_B - \nu\theta_A}{(1-\nu)c} < \frac{\theta_A}{c} \Leftrightarrow d_B - \nu\theta_A < (1-\nu)\theta_A \Leftrightarrow d_B < \theta_A,$$

  which holds by assumption.

- Plugging these expressions for q_A^{SB} and q_B^{SB} back into (1) and (2), we obtain

$$p_B^{SB} = d_B q_B^{SB} = \frac{d_B^2 - \nu\theta_A d_B}{(1-\nu)c},$$

$$p_A^{SB} = \theta_A q_A^{SB} - (\theta_A - d_B) q_B^{SB} = \frac{\theta_A^2}{c} - \frac{(\theta_A - d_B)[d_B - \nu\theta_A]}{(1-\nu)c}.$$

- We have already seen from above that both consumer groups get strictly positive rents. In addition we can now calculate expressions for these rents (although this is not asked about in the question, so doing this is not required). The rents are

$$\theta_B q_B^{SB} - p_B^{SB} = (\theta_B - d_B) q_B^{SB} = \boxed{\tfrac{(\theta_B - d_B)[d_B - \nu\theta_A]}{(1-\nu)c}}$$

for the B consumers and

$$\theta_A q_A^{SB} - p_A^{SB} = \theta_A q_A^{SB} - \theta_A q_A^{SB} + (\theta_A - d_B)\, q_B^{SB} = \boxed{\frac{(\theta_A - d_B)[d_B - \nu\theta_A]}{(1-\nu)c}}$$

for the A consumers.

- Finally we must check that IC-B is satisfied at the optimum:

$$\theta_B q_B^{SB} - p_B^{SB} \geq \theta_B q_A^{SB} - p_A^{SB}$$

or

$$\begin{aligned} (\theta_B - d_B)\, q_B^{SB} &\geq \theta_B q_A^{SB} - \theta_A q_A^{SB} + (\theta_A - d_B)\, q_B^{SB} \\ 0 &\geq (\theta_B - \theta_A)\, q_A^{SB} + (\theta_A - \theta_B)\, q_B^{SB} \\ 0 &\geq -(\theta_A - \theta_B)\left(q_A^{SB} - q_B^{SB}\right), \end{aligned}$$

which is fine thanks to $q_A^{SB} > q_B^{SB}$ (shown above) and the assumption that $\theta_A > \theta_B$.

- **Intuition:** Key to the results is that the A type is the one who gets, for any given q, both: (i) the highest *marginal* utility [the "single-crossing condition"] and (ii) the highest *total* utility.
- Because of (ii), the firm primarily wants to extract the A type's surplus (as it's larger).
- However, if the A type gets too little, he can choose the B type's bundle instead.
  - To prevent this, the monopolist makes the B type's bundle less attractive by offering those consumers less.
  - This works because of (i): The high type suffers more from a reduction in q than the low type.
- Suppose q stands for quality and the firm is a railway company.
  - Then the difference in service level between first- and second-class is larger under second best than under first best:

$$\boxed{\underbrace{\underline{q}^{SB} < \overset{\text{q-distance under FB}}{\overleftrightarrow{\underline{q}^{FB} < \overline{q}^{SB} = \overline{q}^{FB}}}}_{\longleftrightarrow\ \text{q-distance under SB}}}$$

  - The first-class service level is the same under first and second best, whereas the second-class service level is distorted downwards.
  - The intuitive reason: The intended first-class passengers mustn't want to buy second-class tickets instead, so let's make second class sufficiently uncomfortable!

Question 2

Prometheus Sørensen (the principal, P for short) owns a factory producing pencils and wants to hire Absalon Nielsen (the agent, A for short) to work there. If hired, A's task will be to operate a pencil machine and to make sure it runs smoothly. To do this well, A must "make an effort", which involves a (personal) cost to A. This is modelled as A's choosing an effort level $e \in [0,1]$. The associated cost equals $\psi(e)$, where this function satisfies

$$\psi' > 0, \quad \psi'' > 0, \quad \psi(0) = \psi'(0) = 0, \quad \lim_{e \to 1} \psi'(e) = \infty.$$

The number of pencils that come out of the machine, q, is either large ($q = \overline{q}$) or small ($q = \underline{q}$), with $\overline{q} > \underline{q} > 0$. The probability that the number is large equals the effort level: $\Pr(q = \overline{q} \mid e) = e$. P (and the court) can observe which quantity that is realized ($\overline{q}$ or $\underline{q}$) but not the effort level chosen by A. It is assumed that P has all the bargaining power and makes a take-it-or-leave-it offer to A. A contract can specify two numbers, $\overline{t}$ and $\underline{t}$, where $\overline{t}$ is the payment to P if $q = \overline{q}$, and $\underline{t}$ is the payment to A if $q = \underline{q}$. P is risk neutral and his payoff, given a quantity q and a payment t, equals

$$V = q - t.$$

A is also risk neutral and his payoff, given a payment t and an effort level e, equals

$$U = t - \psi(e).$$

A is protected by limited liability, meaning that $\overline{t} \geq 0$ and $\underline{t} \geq 0$. A's outside option would yield the payoff zero.

a) **Let the first best effort level be defined as the one that maximizes the expected total surplus,**

$$(1-e)\underline{q} + e\overline{q} - \psi(e).$$

Characterize this first best level. Explain the economic intuition behind your result.

- Under first best we want to maximize the "pie", and we don't care about how the utilities are distributed across the economic agents. Thus, the choice of the transfers $\overline{t}$ and $\underline{t}$ do not matter (as long as they satisfy the constraints). However, the effort should maximize the total amount of resources available. Therefore, e must be such that the marginal social benefit of increasing e, which is $\overline{q} - \underline{q}$, is equal to the marginal social cost of increasing e, which is $\psi'(e)$. Therefore, the first best solution is characterized by $\overline{q} - \underline{q} = \psi'\left(e^{FB}\right)$.

- The easiest way of solving the problem formally is to maximize the sum of the principal's and the agent's expected payoffs:

$$\begin{aligned} & e\left(\overline{q}-\overline{t}\right)+(1-e)\left(\underline{q}-\underline{t}\right)+e\overline{t}+(1-e)\underline{t}-\psi(e) \\ = \; & e\overline{q}+(1-e)\underline{q}-\psi(e). \end{aligned}$$

  The FOC:

$$\overline{q}-\underline{q}-\psi'\left(e^{FB}\right)=0,$$

  which yields the result.

b) **Derive the second-best solution; that is, characterize the optimal menu of contracts under the assumption that P (and the court) cannot observe the effort level e. Explain the economic intuition behind your result.**

- The principal's problem:

$$\max_{e,\overline{t},\underline{t}}\left\{e\left(\overline{q}-\overline{t}\right)+(1-e)\left(\underline{q}-\underline{t}\right)\right\} \qquad \text{s. t.}$$

$$e\overline{t}+(1-e)\underline{t}-\psi(e)\geq 0 \tag{IR}$$

$$e\in\arg\max_{e'\in[0,1]}\left\{e'\overline{t}+(1-e')\underline{t}-\psi(e')\right\} \tag{IC}$$

$$\overline{t}\geq 0 \qquad \text{and} \qquad \underline{t}\geq 0 \tag{LL}$$

- We now make use of the first-order approach. That is, *we replace the infinitely many IC constraints with the (single) first-order condition of the agent's problem.* The agent's problem if facing a contract $\left(\overline{t},\underline{t}\right)$:

$$\max_{e\in[0,1]}\left\{e\overline{t}+(1-e)\underline{t}-\psi(e)\right\}.$$

  The first-order condition:

$$\overline{t}-\underline{t}=\psi'(e).$$

  Given our assumptions about $\psi(e)$ and as long as $\overline{t}>\underline{t}$, the FOC *must* define the optimal e (the FOC is not only necessary but also sufficient). If $\overline{t}\leq\underline{t}$, the problem would have a corner solution: $e=0$. That is, in this special case with a two-output-level model, we can easily find a condition (namely, $\overline{t}>\underline{t}$) for when the FOC gives us the *global* optimum. We can deal with the case $\overline{t}\leq\underline{t}$ separately, by checking later whether the principal would benefit from inducing $e=0$.

- By replacing the infinitely many IC constraints with the first-order condition, the principal's problem becomes:

$$\max_{e,\overline{t},\underline{t}}\left\{e\left(\overline{q}-\overline{t}\right)+(1-e)\left(\underline{q}-\underline{t}\right)\right\} \qquad \text{s. t.}$$

$$e\bar{t} + (1-e)\,\underline{t} - \psi(e) \geq 0 \tag{IR}$$

$$\bar{t} - \underline{t} = \psi'(e) \tag{IC}$$

$$\bar{t} \geq 0 \qquad \text{and} \qquad \underline{t} \geq 0 \tag{LL}$$

- We can further simplify the principal's problem as follows:
  - (a) Note that, under our assumptions, if e is optimally chosen by the agent, then IR is satisfied. For, given LL, the agent can guarantee himself the utility level zero by choosing $e = 0$.
  - (b) Eliminate $\bar{t}$ from the objective function by plugging in IC:

$$\begin{aligned} V &= e\left(\bar{q} - \bar{t}\right) + (1-e)\left(\underline{q} - \underline{t}\right) \\ &= e\bar{q} + (1-e)\,\underline{q} - e\underbrace{\left(\bar{t} - \underline{t}\right)}_{=\psi'(e) \text{ by IC}} - \underline{t} \\ &= e\bar{q} + (1-e)\,\underline{q} - e\psi'(e) - \underline{t} \end{aligned}$$

  - (c) Note that since V is decreasing in $\underline{t}$, the LL constraint in a bad state must bind: $\underline{t} = 0$. Hence the principal's objective function can be written as

$$V = e\bar{q} + (1-e)\,\underline{q} - e\psi'(e).$$

- The principal's problem can now be stated as follows, without any constraints and with only one choice variable, e:

$$\max_e \left\{e\bar{q} + (1-e)\,\underline{q} - e\psi'(e)\right\}.$$

  The first-order condition:

$$\bar{q} - \underline{q} - \psi'\left(e^{SB}\right) - e^{SB}\psi''\left(e^{SB}\right) = 0. \tag{*}$$

  The second-order condition:

$$-2\psi''(e) - e\psi'''(e) < 0 \qquad \text{for all } e \in [0,1].$$

  This SOC is not automatically satisfied — we need to assume that the function $\psi(e)$ is such that it is. A sufficient condition, given our other assumptions about function $\psi(e)$, is that its third derivative is positive.

- However, before concluding that e^{SB} is the optimum, we should check that the principal cannot be made better off by choosing $e = 0$ (recall that this was the possibility we ignored when plugging in the FOC). By setting $\bar{t} = \underline{t} = 0$, which would induce $e = 0$, the principal would get the expected utility:

$$V = e\left(\bar{q} - \bar{t}\right) + (1-e)\left(\underline{q} - \underline{t}\right) = \underline{q}.$$

From above we know that V can be written as $V = e\overline{q} + (1-e)\,\underline{q} - e\psi'(e)$. Hence,

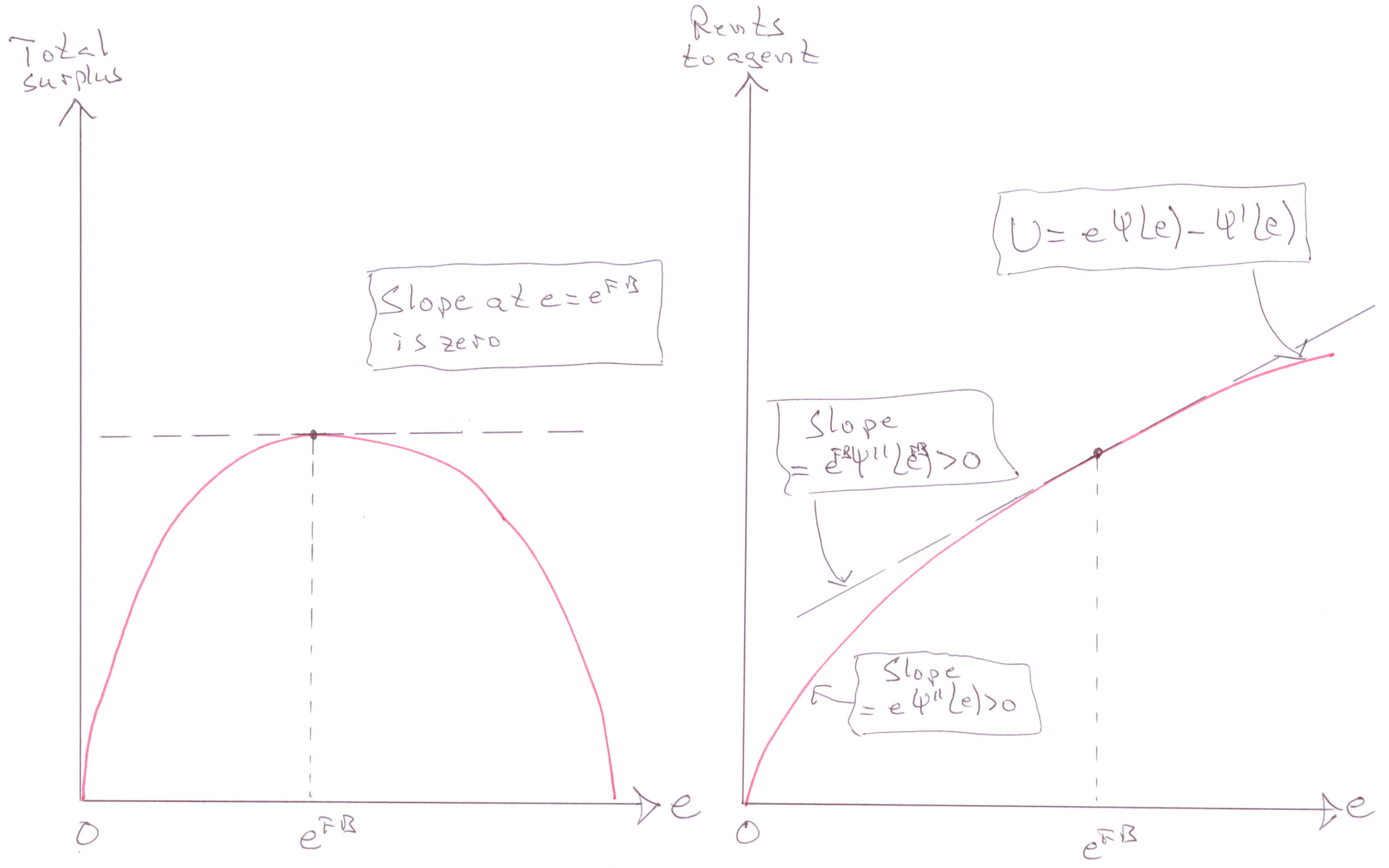
$$\begin{aligned} V^{SB} &= e^{SB}\overline{q} + \left(1-e^{SB}\right)\underline{q} - e^{SB}\psi'\left(e^{SB}\right) \\ &= \underline{q} + e^{SB}\underbrace{\left[\overline{q} - \underline{q} - \psi'\left(e^{SB}\right)\right]}_{=e^{SB}\psi''\left(e^{SB}\right)\text{ by (*)}} \\ &= \underline{q} + \left(e^{SB}\right)^2\psi''\left(e^{SB}\right) > \underline{q}. \end{aligned}$$

We can conclude that $e = e^{SB}$ is indeed the optimum.

- In summary:
  - The second best effort level is defined by
  $$\overline{q} - \underline{q} = \psi'\left(e^{SB}\right) + e^{SB}\psi''\left(e^{SB}\right). \tag{*}$$
  - The second best transfer in a good state is given by the agent's FOC above:
  $$\overline{t}^{SB} = \psi'\left(e^{SB}\right) = \overline{q} - \underline{q} - e^{SB}\psi''\left(e^{SB}\right) > 0.$$
  - The second best transfer in a bad state is given by the binding LL constraint:
  $$\underline{t}^{SB} = 0.$$
- **The intuition** is similar to the one in adverse selection models: By lowering e a bit from the first best level, the principal saves more on transfers than he loses in terms of efficiency.
- Exactly as in the adverse selection model, the principal faces a trade-off between efficiency and rent extraction:

$$\boxed{V = \underbrace{\left[e\overline{q} + (1-e)\,\underline{q} - \psi(e)\right]}_{=\text{Total surplus}} - \underbrace{\left[e\overline{t} + (1-e)\,\underline{t} - \psi(e)\right]}_{=U\text{, the agent's rent}}.}$$

- The principal could — if he wanted to — induce the agent to exert the first-best level of effort.
  - He would simply choose transfers that satisfy $\overline{t} - \underline{t} = \psi'\left(e^{FB}\right)$.
  - The **benefit**: it would maximize the size of the pie.
  - The **cost:** the principal must pay relatively large transfers.
- However, at $e = e^{FB}$ the marginal effect of a change in e on the size of the pie is zero, whereas the marginal effect of a change in e on the size of the transfer is strictly positive. See also figure (L9-I, fig 4, v2).
  - Hence it always pays off to reduce e at least a little bit.


Total surplus
Slope at e = e^FB
is zero
0
e^FB
e
Rents to agent
U = eψ(e) − ψ'(e)
Slope
= e^FB ψ''(e^FB) > 0
Slope
= eψ''(e) > 0
0
e^FB
e

29-I, fig 4